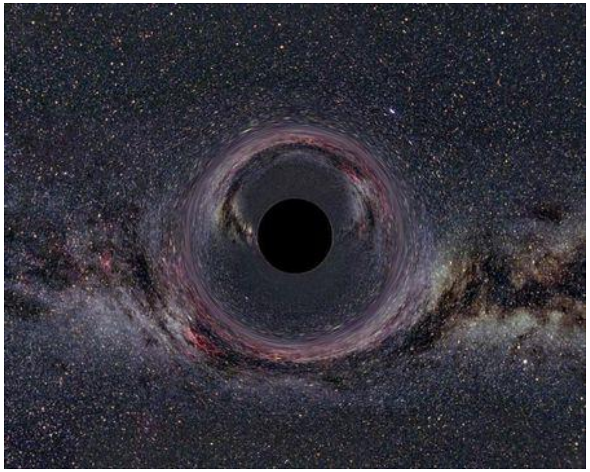
Strong energy condition and holographic complexity growth bound



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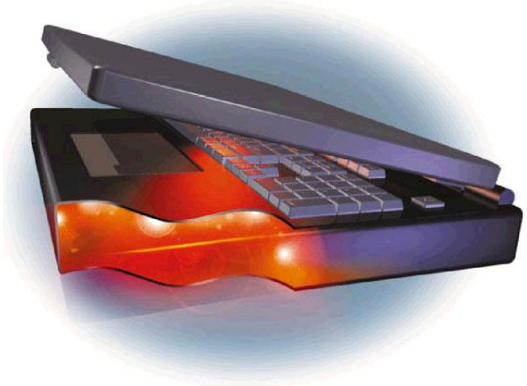


Outline

- Complexity and its growth rate bound
- Complexity-action conjecture
- Energy condition and action growth rate bound

Based on:

[1] A. R. Brown, et. al, Phys. Rev. Lett. 116, 191301 (2016);
[2] A. R. Brown, et. al, Phys. Rev. D 93, 086006 (2016);
[3] L. Lehner, et. al., Phys. Rev. D 94, 084046 (2016);
[4] R.-Q. Yang, arXiv: 1610.05090 [gr-qc].

What is complexity?

- Quantum complexity is a quantum information quantity which has to do with the difficulty converting one quantum state to another quantum state.

- Consider a state : $|00\rangle$. How many steps we need at least to change it into an other state $|11\rangle$?

- The complexity of this state is 2.

- Any computation task can be regarded as a transition from one state to an other state;

$$|00100110100\cdots\rangle \longrightarrow |10101100100\cdots\rangle$$

Complexity can describe the complexity of any computation task

- Computers are physical systems: the laws of physics dictate what they can and cannot do.
- For a compute with energy(mass) E, the times it can compute in per second is,

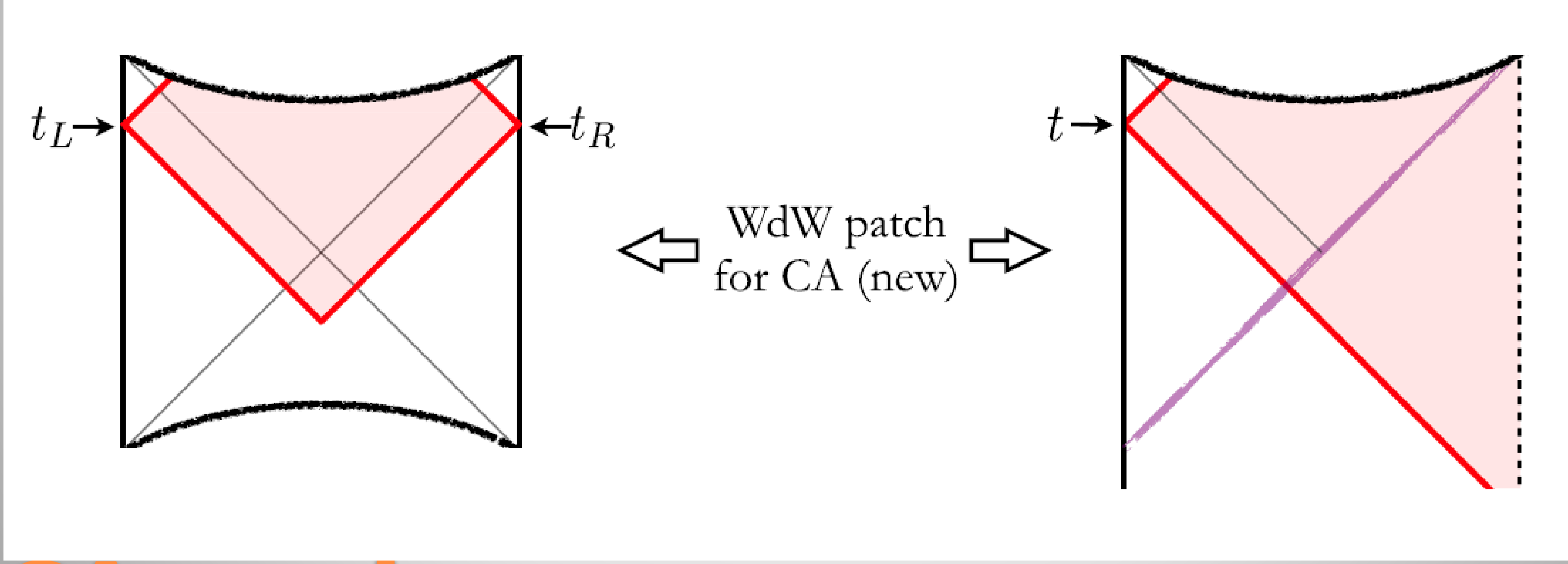
$$\frac{\mathrm{d}C}{\mathrm{d}t} \leq \frac{2E}{\pi\hbar}$$

- For a compute with mass=1kg, the fastest computational ability is about 5×10^{50} /sec
- Is there any real system can reach this up bound?

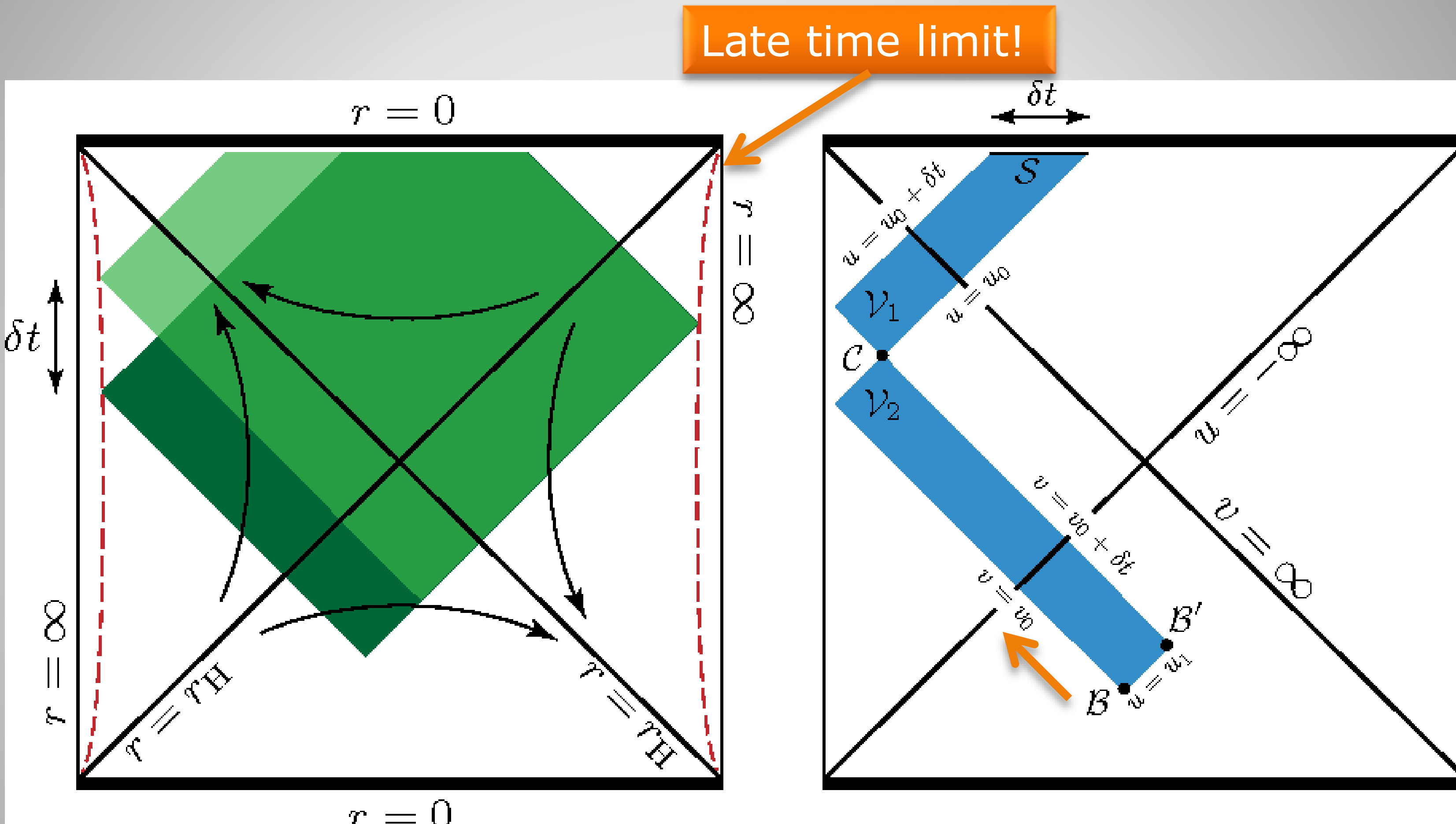
How fast a compute can be?

- Complexity-action (AC) states that the complexity is given by the bulk action evaluated on the Wheeler-deWitt patch attached at some boundary time t

$$\mathcal{C} = \frac{\mathcal{A}}{\pi \hbar}$$



CA conjecure



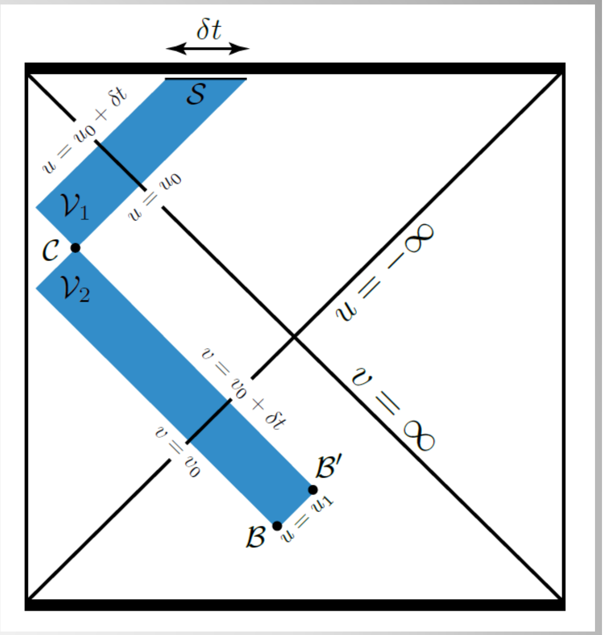
Action growth in Schwarzschild AdS black hole

- The variation of complexity after a time δt can be expressed as,

$$\delta S = S_{\mathcal{V}_1} - S_{\mathcal{V}_2} - 2\int_{\mathcal{S}} K\, d\Sigma + 2\oint_{\mathcal{B}'} a\, dS - 2\oint_{\mathcal{B}} a\, dS,$$

- After some computations, we can find

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = 2M}$$



Action growth Schwarzschild AdS black hole

Complexity growth rate bound

Quantum machines gives following inequality

$$\frac{\mathrm{d}C}{\mathrm{d}t} \le \frac{2E}{\pi\hbar}$$

CA conjecture shows that the complexity is given by bulk action

$$C = \frac{A}{\pi\hbar}$$

In asymptotic AdS black hole, the E is just the mass of black hole

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} \le 2M}$$

In late time limit, Schwarzschild AdS black hole saturates this inequality!

Analogue with black hole thermodynamics

Conjectures by quantum gravity	Thermodynamic results	Classical version
	T is uniform in static case	κ is uniform in event horizon
$T=\kappa/2\pi$	T can't reduce to zero by finite operators	κ can't reduce to zero with finite advanced time
$S=A/2\pi$	$\delta S\geq 0$	$\delta A\geq 0$
$U=M$	$\delta U=T\delta S+F^i\,\delta x_i$	$\delta M=\kappa\delta S/8\pi+F^i\,\delta x_i$

Can a similar thing happen in CA conjecture?

$$\boxed{\frac{\mathrm{d}C}{\mathrm{d}t}\leq\frac{2E}{\pi\hbar}} \xrightarrow[C=\frac{A}{\pi\hbar}]{\text{Classical version}} \boxed{\frac{\mathrm{d}A}{\mathrm{d}t}\leq 2M}$$

$$\boxed{\frac{\mathrm{d}C}{\mathrm{d}t} \le \frac{2E}{\pi\hbar}} \xrightarrow[C = \frac{A}{\pi\hbar}]{\text{Classical version}} \boxed{\frac{\mathrm{d}A}{\mathrm{d}t} \le 2M}$$

- In the late time limit, can we always find that this result?

- Dose the saturation appear only when space-time is a vacuum black hole?

Is it a universal result?

Bound equation

- In order to answer this question, let's first pay more attention on the bound equation it self

Has proper definition

Coordinate dependent

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} \le 2M}$$

Has several different definitions such as:
ADM mass, Misner-Sharp mass, Komar mass, Brown-York tensor an so on.

Coordinate dependent

In static case, a good choice for t is that it is the orbit of Killing vector

What's the meaning of mass in this inequality?

- In quantum field theory, as there is a global Poincare group, mass is a Casimir's invariants of Poincare group;
- In GR, the absence of global Poincare group leads the energy of gravitational field is nonlocal;
- In non-isolated system, the total energy is ill-defined.

Mass ambiguous in GR

- To avoid the difficult in the definition about mass, we only consider the isolated system, i.e., when r→∞,

$$|r^3 T^{\mu}{}_{\nu}| < \infty$$

- Define a Komar's mass for any closed two surface:

Killing potential:

$$\xi^{\mu} = \nabla_{\nu}\omega^{\nu\mu}$$

$$m = -\frac{1}{8\pi}\oint_S \left(\nabla^{\mu}\xi^{\nu} - \frac{3}{\ell^2_{\mathrm{AdS}}}\omega^{\mu\nu}\right)\mathrm{d}S_{\mu\nu}$$

- The total mass M then can be obtained by setting S→∞,

$$\boxed{M = -\frac{1}{8\pi}\oint_{S\to\infty}\left(\nabla^{\mu}\xi^{\nu} - \frac{3}{l^2_{\mathrm{AdS}}}\omega^{\mu\nu}\right)\mathrm{d}S_{\mu\nu}}$$

Geometrical formulation

- Once fixed the tR, the action in WdW patch can be treated a s a scalar field on boundary;

- As Killing vector ξ^{μ} lays on the boundary, it can be seen as a vector field at the boundary, it's well defined that,

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = \mathcal{L}_{\xi} A}$$

An inequality only involves Killing vector and on-shell action.

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} \leq 2M} \Longrightarrow \boxed{\mathcal{L}_{\xi} A \leq -\frac{1}{4\pi} \oint_{S\to\infty} \left(\nabla^{\mu} \xi^{\nu} - \frac{3}{l_{\mathrm{AdS}}^{2}} \omega^{\mu\nu} \right) \mathrm{d}S_{\mu\nu}}$$

PROVE IT

For a static asymptotic isolated AdS black hole (maximal extension), if it satisfies that,

- It has topology $R^2 \times S^2$ and only space-like singularities;
- There is a connected bifurcate Killing horizon;
- Matter fields only distribute in the outside of the Killing horizon;
- Strong energy condition (SEC) is satisfied.

Then at late time limit, this inequality is true,

$$\frac{\mathrm{d}A}{\mathrm{d}t} \leq 2M$$

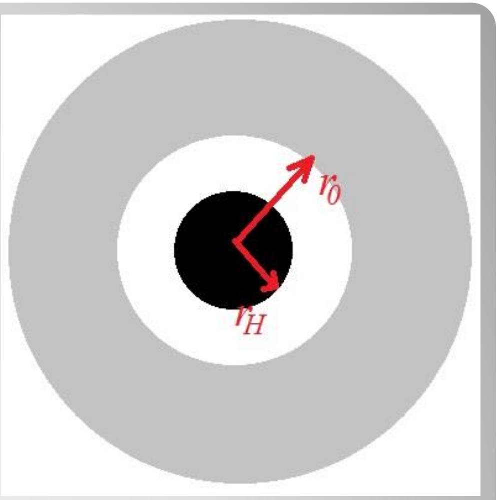
The result

Table 2.1: Energy conditions.

Name	Statement	Conditions
Weak	$T_{\alpha\beta}v^\alpha v^\beta \geq 0$	$\rho \geq 0, \quad \rho + p_i > 0$
Null	$T_{\alpha\beta}k^\alpha k^\beta \geq 0$	$\rho + p_i \geq 0$
Strong	$(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta})v^\alpha v^\beta \geq 0$	$\rho + \sum_i p_i \geq 0, \quad \rho + p_i \geq 0$
Dominant	$-T^\alpha_\beta v^\beta$ future directed	$\rho \geq 0, \quad \rho \geq \lvert p_i \rvert$

Local energy conditions

- let's consider the case there is a nonzero energy momentum tensor field $T^{\mu\nu}$ at the region of $r > r_0$
- One can find that $dA/dt=2m_H$.



- But by Einstein equation, one can find,

$$m = -\frac{1}{8\pi}\oint_S \left(\nabla^\mu \xi^\nu - \frac{3}{\ell^2_{\mathrm{AdS}}}\omega^{\mu\nu}\right)\mathrm{d}S_{\mu\nu}$$
$$= 2\int_\Sigma (T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})n^\mu\xi^\nu \mathrm{d}V$$

$$n^\mu = \xi^\mu/\sqrt{-\xi^\alpha\xi_\alpha}$$

$$m(r) = m_H + 2\int_{\Sigma(r>r_0)}(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})n^\mu\xi^\nu \mathrm{d}V$$

Why we need SEC?

Why we need SEC?

- In some region of Σ where the pressure is too negative so that

$$\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)\xi^{\mu}\xi^{\nu} < 0$$

$$m(r) = m_H + 2\int_{\Sigma(r>r_0)} (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) n^{\mu}\xi^{\nu} \mathrm{d}V$$

one may obtain $M = m(\infty) < m_H$;

- In order to insure the inequality, we need that $M \geq m_H$.
- One sufficient condition is SEC.

$$\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)\xi^{\mu}\xi^{\nu} \geq 0$$

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} \leq -\frac{1}{4\pi}\oint_{S\to\infty}\left(\nabla^{\mu}\xi^{\nu} - \frac{3}{l_{\mathrm{AdS}}^{2}}\omega^{\mu\nu}\right)\mathrm{d}S_{\mu\nu}}$$

- As the space-time is static, the action growth in the outside of horizon is canceled with each other

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_{\mathrm{in,bulk}} + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_{\mathrm{in,bd}} + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_{\mathrm{in,corner}}$$

The main idea of proof

$$\frac{\mathrm{d}A_{in,bulk}}{\mathrm{d}t} = -\frac{3}{8\pi l_{AdS}^2}\int_{\Sigma'}\sqrt{\xi^\mu \xi_\mu}\,\mathrm{d}V$$

$$\frac{\mathrm{d}A_{in,bd}}{\mathrm{d}t} = \frac{3}{2}m_H$$

$$\frac{\mathrm{d}A_{in,corner}}{\mathrm{d}t} = \frac{1}{2}m_H + \frac{3}{8\pi l_{AdS}^2}\int_{\Sigma'}\sqrt{\xi^\mu \xi_\mu}\,\mathrm{d}V$$

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A_{in,bulk}}{\mathrm{d}t} + \frac{\mathrm{d}A_{in,bd}}{\mathrm{d}t} + \frac{\mathrm{d}A_{in,corner}}{\mathrm{d}t} = 2m_H}$$

- Now use the SEC, we see

$$M = m_H + \int_\Sigma \left(T^{\mu\nu} - \frac{T}{2}g^{\mu\nu}\right) n_\mu \xi_\nu dV \geq m_H$$

- Finally, we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} \leq 2M$$

The proof

- Although the inequality is proposed by the consideration of CA conjecture, the proof of doesn't rely on the correctness of CA conjecture and holographic duality.
- It is a very strong evidence for CA conjecture!
- It gives us a new viewpoint to see connection between classical gravity and quantum information theory.

Discussion

- There are still some very important cases that can't be covered.
- The most important one is Kerr-Newman black hole, where Maxwell field extends into the horizon and time-like singularities appear.
- It is also interesting to investigate if they could be weakened.

Further works